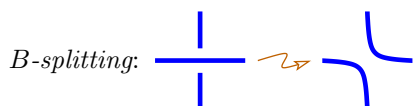
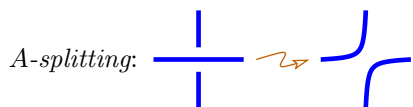


The Kauffman bracket and the Jones polynomial [Ka1]

Let L be a link diagram.



A state s is a choice of either A - or B -splitting at every classical crossing.

$\alpha(s)$ = # (of A -splittings in s)

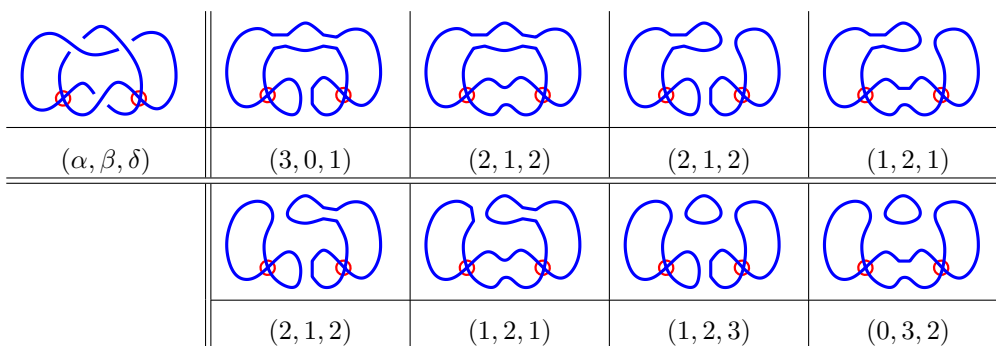
$\beta(s)$ = # (of B -splittings in s)

$\delta(s)$ = # (of circles in s)

$$[L](A, B, d) := \sum_{s \in \mathcal{S}(L)} A^{\alpha(s)} B^{\beta(s)} d^{\delta(s)-1}$$

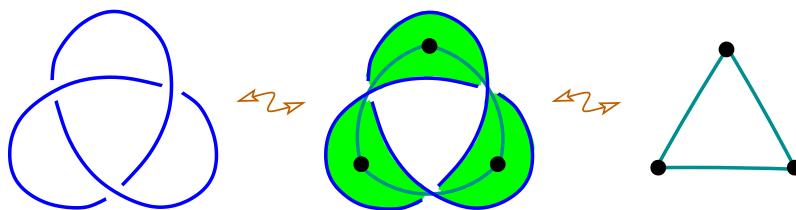
$$J_L(t) := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example



$$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \quad J_L(t) = 1$$

Thistlethwaite's Theorem [Ka1] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t, -t^{-1})$.*

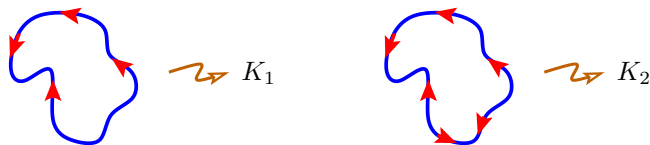


Arrow polynomial [DK]

If a splitting does not respect the orientation, we put two arrows on the branches of the splitting oriented counterclockwise near the crossing:



Cancellation of arrows. All pairs of arrows which are next to each other on a state circle and pointing in the same direction are canceled.



With each state circle c we associate the variable $K_{i(c)}$ as follows. The $i(c)$ is equal to half of the number of arrows along c remained after the cancellations. We set $K_0 = 1$ and define the *arrow bracket polynomial* as

$$[L]_A(A, B, d, K_i) := \sum_{s \in \mathcal{S}(L)} A^{\alpha(s)} B^{\beta(s)} d^{\delta(s)-1} \prod_{c \in s} K_{i(c)} .$$

Thus the arrow bracket polynomial $[L]$ becomes a polynomial in infinitely many variables A, B, d, K_1, K_2, \dots . However, for a concrete diagram L only finitely many K 's appear in it.

The standard substitution $B := A^{-1}$, $d := -A^2 - A^{-2}$ gives the *normalized Dye-Kauffman arrow polynomial* [DK]:

$$\langle L \rangle_{NA} := (-A^3)^{-w(L)} [L]_A(A, A^{-1}, -A^2 - A^{-2}, K_i) ,$$

which is an invariant of virtual links. The invariance under the Reidemeister moves follows from the cancellation rule. A remarkable observation of H. Dye and L. Kauffman is that for classical link diagrams, all arrows will be canceled out and the K 's variables do not occur in the arrow polynomial. In this case it is essentially equivalent to the Jones polynomial (after the further substitution $A = t^{-1/4}$).

Arrow Thistlethwaite's theorem is proved in [BBC].

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- [DK] H. Dye, L. Kauffman, *Virtual Crossing Number and the Arrow Polynomial*, J. Knot Theory Ramifications, **18**(10)(2009) 1335–1357. Preprint [arXiv:math.GT/0810.3858](https://arxiv.org/abs/math/0810.3858)
- [Ka1] L. H. Kauffman, *New invariants in knot theory*, Amer. Math. Monthly **95** (1988) 195–242.
- [Lik] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Math. **175**, Springer-Verlag, New York, 1997.